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CALCULATION OF THE CHARACTERISTICS OF THIN ELASTIC RODS WITH A PERIODIC STRUCTURE*

A.G. KOLPAKOV

A transfer is made from the three-dimensional theory of elasticity in a thin rod to the problem of the theory of beams. The transfer differs from the problems of plates discussed in /1, 2/ (see also the references in /3/), in that the dimensions are reduced, during the passage to the limit, by two, i.e. from three (the dimensions of the initial problem in the theory of elasticity) to one (the dimensions of the limit problem in the theory of beams). In /1, 2/ the dimensions change from three to two. This leads to differences in the form of the asymptotic expansion, in the methods for studying it and in the appearance of new types of cellular problems.

1. Formulation of the problem. Let us consider the region Ω_ε which has a periodic structure, where the diameter of the transverse cross-section and the size of the periodicity cell (PC) P_ε are of the order of $\varepsilon \ll 1$ (Fig.1). When $\varepsilon \to 0$, the region contracts to the segment $[-1, 1]$. Let the region be occupied by elastic material whose tensor of elastic constants will be denoted by $a_{ijkl}(\mathbf{x}/\varepsilon)$ and regarded as its periodic function with PC P_ε. The equations of equilibrium for this body have the form

$$\int_{\Omega_\varepsilon} \sigma_{ij} v_{i,j}\, dv + \varepsilon^a \int_{\Gamma_\varepsilon} \mathbf{g}\mathbf{v}\, ds = \varepsilon^b \int_{\Omega_\varepsilon} \mathbf{f}\mathbf{v}\, dv \tag{1.1}$$

$$\forall \mathbf{v} \in V(\Omega_\varepsilon) = \{\mathbf{v} \in H^1(\Omega_\varepsilon):\ \mathbf{v}(\mathbf{x}) = 0 \text{ when } x_1 = \pm 1\}$$

(the functional class H^1 is defined in /4/). Here σ_{ij} are local stresses connected with the local displacements $\mathbf{u} \in V(\Omega_\varepsilon)$ by Hooke's law

$$\sigma_{ij} = \varepsilon^{-4} a_{ijkl}(\mathbf{x}/\varepsilon)\, u_{k,l} \tag{1.2}$$

Note 1. The presence of a multiplier ε^{-4} in Hooke's law is connected with the known estimate of the order of the moments of inertia /5/. In the case of plates in the same situation, the order was taken into account in /1/ by introducing the multiplier ε^{-3}.

Note 2. The power indices a and b in (1.1) take into account the orders of the volume and surface forces, and their values will be chosen later.

2. Asymptotic expansions. We shall seek a formal asymptotic expansion in the form

$$\mathbf{u} = \mathbf{u}^{(0)}(x_1) + \varepsilon \mathbf{u}^{(1)}(x_1, \mathbf{y}) + \ldots = \sum_{k=0}^{\infty} \varepsilon^k \mathbf{u}^{(k)} \tag{2.1}$$

$$\mathbf{v} = \mathbf{v}^{(0)}(x_1) + \varepsilon \mathbf{v}^{(1)}(x_1, \mathbf{y}) + \ldots = \sum_{k=0}^{\infty} \varepsilon^k \mathbf{v}^{(k)}$$

$$\sigma_{ij} = \varepsilon^{-4}\sigma_{ij}^{(-4)}(x_1, \mathbf{y}) + \varepsilon^{-3}\sigma_{ij}^{(-3)}(x_1, \mathbf{y}) + \ldots = \sum_{m=-4}^{\infty} \varepsilon^m \sigma_{ij}^{(m)}$$

where $x_1 \in [-1, 1]$ is the slow variable and $\mathbf{y} = \mathbf{x}/\varepsilon$ the fast variable /6/. The functions appearing in the expansion (2.1) are assumed to be periodic in y_1, with PC $P_1 = \varepsilon^{-1} P_\varepsilon$. Differentiation operators for the functions of the arguments x_1, y have the form /6/

$$\frac{\partial}{\partial x_1} \to \frac{\partial}{\partial x_1} + \varepsilon^{-1}\frac{\partial}{\partial y_1},\quad \frac{\partial}{\partial x_\alpha} \to \varepsilon^{-1}\frac{\partial}{\partial y_\alpha} \quad (\alpha = 2, 3) \tag{2.2}$$

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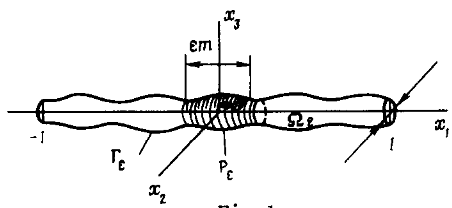

Fig.1

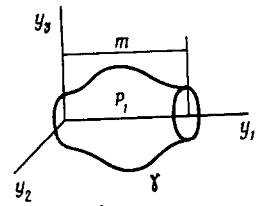

Fig.2

In what follows, we shall use Greek letters to denote the indices which take the values 2 and 3, and Latin letters for indices which take the values 1, 2, 3.

Let us now change from problem (1.1) in the variable (depending on ε) region Ω_ε, to the problem in a fixed region. To do this we make the following change of variables:

$$x_1 \to x_1,\ x_\alpha \to y_\alpha = \varepsilon^{-1}x_\alpha \quad (\alpha = 2, 3) \tag{2.3}$$

transforming the Ω_ε to the region Ω_1 of fixed size. The same statement applies to the side surface Ω_ε, i.e. to the surface Γ_ε. After the change of variable (2.3) and after writing the derivatives for the functions of the arguments $x_1, \mathbf{y}$ in the form (2.2), Eq.(1.1) becomes

$$\varepsilon^2 \int\limits_{\Omega_1} \left(\sigma_{ij}\varepsilon^{-1}\frac{\partial v_i}{\partial y_j} + \sigma_{i1}\frac{\partial v_i}{\partial x_1}\right) dv + \varepsilon^{2+a}\int\limits_{\Gamma_1} \mathbf{g}\mathbf{v}\, ds = \varepsilon^{2+b}\int\limits_{\Omega_1} \mathbf{f}\mathbf{v}\, dv \tag{2.4}$$

and $\forall \mathbf{v} \in V(\Omega_1)$, where the functional class $V(\Omega_1)$ is defined in the same manner as $V(\Omega_\varepsilon)$.

In Eqs.(1.1) and (2.4) dv, ds are the measures on the corresponding sets and in the corresponding variables.

In the variables $\mathbf{y} = \mathbf{x}/\varepsilon$, the PC P_ε becomes $P_1 = \varepsilon^{-1}P_\varepsilon = \{\mathbf{y} = \mathbf{x}/\varepsilon\colon\ \mathbf{x} \in P_\varepsilon\}$. In what follows, we shall use $\langle\cdot\rangle = m^{-1}\int\limits_{P_1} \cdot\, d\mathbf{y}$ averaged over PC and the known dependence of the integrals on the functions periodic in $\mathbf{y} = \mathbf{x}/\varepsilon$, with the integral of their mean (see e.g. /1/)

$$\lim_{\varepsilon\to 0} \int\limits_{\Omega_1} f(x_1, \mathbf{x}/\varepsilon)\, d\mathbf{x} = \int\limits_{-1}^{1} \langle f\rangle (x_1)\, dx_1$$

Let us substitute the expansion (2.1) into (2.4). We obtain

$$\sum_{k=0}^{\infty}\sum_{m=-4}^{\infty} \varepsilon^2 \int\limits_{\Omega_1} (\varepsilon^{m+k+1}\sigma_{ij}^{(m)}v_{i,\,jy}^{(k)} + \varepsilon^{m+k}\sigma_{i1}^{(m)}v_{i,\,1x}^{(k)})\, dv + \sum_{k=0}^{\infty}\varepsilon^{2+a}\int\limits_{\Gamma_1} \varepsilon^k \mathbf{g}\mathbf{v}^{(k)}\, ds = \sum_{k=0}^{\infty}\varepsilon^{2+b}\int\limits_{\Omega_1}\varepsilon^k\mathbf{f}\mathbf{v}^{(k)}\, dv \tag{2.5}$$

Here and henceforth jy will denote $\partial/\partial y_j$ and $1x$ is $\partial/\partial x_1$. Substituting the expansion (2.1) into Hooke's law (1.2) and equating terms of like powers in ε, we obtain

$$\sigma_{ij}^{(m)} = a_{ijkl}(\mathbf{y})\, u_{k,\,ly}^{(m+5)} + a_{ijk1}(\mathbf{y})\, u_{k,\,1x}^{(m+4)} \tag{2.6}$$

Here and henceforth $m = -4, -3, \ldots;\ k = 0, 1, \ldots.$

3. ***Derivation of the system of equations of the theory of beams.*** We shall next discuss problem (2.5) for various values of m, k and for a suitable choice of the test function $\mathbf{v}$.

1°. Putting $k = 0$ we obtain, respectively, $\mathbf{v} = \mathbf{v}^{(0)}(x_1)$ and $v_{i,\,jy} = 0$. As a result we obtain from (2.5), for this case,

$$\sum_{m=-4}^{\infty} \varepsilon^2 \int\limits_{\Omega_1} \varepsilon^m\sigma_{i1}^{(m)}v_{i,\,1x}^{(0)}\, dv + \varepsilon^{2+a}\int\limits_{\Gamma_1}\mathbf{g}\mathbf{v}^{(0)}\, ds = \sum_{m=-4}^{\infty}\varepsilon^{2+b}\int\limits_{\Omega_1}\mathbf{f}\mathbf{v}^{(0)}\, dv$$

Equating in this expression terms of like powers of ε we obtain, taking into account the remark concerning the mean values,

$$\langle\sigma_{i1}^{(-4)}\rangle_{,1x} = 0 \ (m = -4)$$
$$\langle\sigma_{i1}^{(-3)}\rangle_{,1x} = 0 \quad (m = -3) \tag{3.1}$$
$$\langle\sigma_{i1}^{(-2)}\rangle_{,1x} = \langle g_i\rangle_\gamma + \langle f_i\rangle \quad (m = -2)$$

Here we have put $a = b = -2$. We denote by $\langle\cdot\rangle_\gamma = m^{-1}\int_\gamma \cdot ds$ the PC P_1 (Fig.2) averaged over the side surface γ.

When $m > -2$, the powers in the equations given above are positive.

2°. Let us put in (2.5) $k = 1$, and take the test function in the form

$$\mathbf{v} = \mathbf{v}^{(1)}(x_1, \mathbf{y}) = y_2\mathbf{v}_2(x_1) + y_3\mathbf{v}_3(x_1) \tag{3.2}$$

With this choice of k and $\mathbf{v}$ relation (2.5) will become

$$\sum_{m=-4}^{\infty} \varepsilon^2 \int_{\Omega_1} [\varepsilon^m\sigma_{ij}^{(m)}(v_{2i}\delta_{2j} + v_{3i}\delta_{3j}) + \varepsilon^{m+1}\sigma_{i1}^{(m)}(y_2v_{2i,1x} + y_3v_{3i,1x})]\,dv + \tag{3.3}$$
$$\int_{\Gamma_1} \mathbf{g}(y_2\mathbf{v}_2 + y_3\mathbf{v}_3)\,d\gamma = \sum_{m=-4}^{\infty}\int_{\Omega_1} \mathbf{f}(y_2\mathbf{v}_2 + y_3\mathbf{v}_3)\,dv$$

Let us bring into our discussion the quantities $M_{\alpha i}^{(m)} = \langle y_\alpha\sigma_{i1}^{(m)}\rangle$, which can be regarded in the mechanical sense, as moments /1/. Taking into account the remarks about the mean values and equating terms of like non-positive powers, we obtain

$$\langle\sigma_{i\alpha}^{(-4)}\rangle = 0 \ (\alpha = 2, 3) \ (m = -4) \tag{3.4}$$
$$-M_{\alpha i,1x}^{(-4)} + \langle\sigma_{i\alpha}^{(-3)}\rangle = 0 \quad (\alpha = 2, 3) \ (m = -3)$$
$$-M_{\alpha i,1x}^{(-3)} + \langle\sigma_{i\alpha}^{(-2)}\rangle = \langle g_iy_\alpha\rangle_\gamma + \langle f_iy_\alpha\rangle \quad (m = -2) \quad (\alpha = 2, 3) \tag{3.5}$$

When $m > -2$, the powers of ε in (3.3) are positive.

3°. Let us write $k = 1, m = -4$ and take the test function in the form $\mathbf{v} = \mathbf{v}^{(1)}(\mathbf{y})$, periodic in y_1, with the PC P_1. In this case $v_{i,1x}^{(1)} = 0$. Under these conditions we obtain, from (2.5),

$$\sigma_{ij,jy}^{(-4)} = 0 \ \text{ and } \ \sigma_{ij}^{(-4)}n_j = 0 \ \text{ on } \Gamma_1 \tag{3.6}$$

where $\mathbf{n}(\mathbf{y})$ is the normal to side surface Γ_1 of the region Ω_1. Let us use relation (2.6) with $m = -4$. Substituting, in accordance with relations shows, $\sigma_{ij}^{(-4)}$ into (3.6), we obtain

$$(a_{ijkl}(\mathbf{y})u_{k,ly}^{(1)} + a_{ijp1}(\mathbf{y})u_{p,1x}^{(0)}(x_1))_{,jy} = 0 \tag{3.7}$$

with boundary condition

$$(a_{ijkl}(\mathbf{y})u_{k,ly}^{(1)} + a_{ijl1}(\mathbf{y})u_{p,1x}^{(0)}(x_1))n_j(\mathbf{y}) = 0 \tag{3.8}$$

and the condition that $\mathbf{u}^{(1)}(\mathbf{y})$ is periodic in y_1 with PC P_1.

We draw attention to the replacement made here (in order to make further entries clearer) of the summation index (the index p).

Eqs.(3.7) and (3.8) lead to the cellular problem (CP). As we known, /1, 2, 6, 7/, the CP plays a fundamental role in determining the mechanical characteristics of materials and constructions with a periodic structure. Let us introduce the functions $\mathbf{X}^{1p}(\mathbf{y})$ as the solution of the "first CP theory of beams":

$$(a_{ijkl}(\mathbf{y})X_{k,ly}^{1p} + a_{ijp1}(\mathbf{y}))_{,jy} = 0 \ \text{ in } P_1 \tag{3.9}$$
$$(a_{ijkl}(\mathbf{y})X_{k,ly}^{1p} + a_{ijp1}(\mathbf{y}))n_j = 0 \ \text{ on } \ \gamma$$

$\mathbf{X}^{1p}(\mathbf{y})$ is periodic in y_1 with PC P_1 (Fig.2).

Comparing problem (3.7), (3.8) with the PC (3.9), we obtain the relation (compare with /1/)

$$\mathbf{u}^{(1)} = \mathbf{X}^{1p}(\mathbf{x}/\varepsilon)u_{p,1x}^{(0)}(x_1) + \mathbf{V}(x_1) \tag{3.10}$$

Note 3. Some of the functions $\mathbf{X}^{1p}$ can be calculated explicitly, namely

$$X_k^{1\alpha}(\mathbf{y}) = -\delta_{1k} y_\alpha \quad (\alpha = 2, 3) \tag{3.11}$$

Verification. $a_{ijkl} X^{1\alpha}_{k,ly} = -a_{ij1\alpha} + a_{ij\alpha 1} = 0$ (the latter with the symmetry of elastic constants tensor taken into account /5/).

The functions $\mathbf{X}^{1p}(\mathbf{y})$ $(p = 1, 2, 3)$ are particular solutions of CP (3.9) for various values of the index p. The homogeneous problem corresponding to (3.9) (the problem is obtained by putting $a_{ijp1}(\mathbf{y}) = 0$ in (3.9)), has the solution $\mathbf{X} = (0, y_3, -y_2)\,\varphi(x_1)$ where $\varphi(x_1)$ is an arbitrary (so far) function of the argument x_1. We note that in case of the problems with the variables $\mathbf{y}$ the functions of the argument x_1 are regarded as constants.

Verification. $a_{ijkl} X_{k,ly} = \varphi(x_1)(a_{ij23} - a_{ij32})$ (the latter by virtue of $a_{ijkl} = a_{ijlk}$ /5/).

From (3.10) and (3.11) we obtain the following expressions (written in coordinate form?:

$$u_1^{(1)} = X_1^{11}(\mathbf{y})\, u^{(0)}_{1,1x}(x_1) - y_\alpha u^{(0)}_{\alpha,1x}(x_1) + V_1(x_1) \tag{3.12}$$

$$u_\beta^{(1)} = X_\beta^{11}(\mathbf{y})\, u^{(0)}_{1,1x} + V_\beta(x_1) + y_{\bar\beta} s_\beta \varphi(x_1)$$

$$(\alpha, \beta = 2, 3), \quad \bar\beta = \begin{cases} 3 \text{ when } \beta = 2, & s_1 = 0,\ s_2 = 1,\ s_3 = -1 \\ 2 \text{ when } \beta = 3, & \end{cases}$$

Substituting (3.12) into (2.6) we obtain, when $m = -4$,

$$\sigma_{ij}^{(-4)} = a_{ij11}(\mathbf{y})\, u^{(0)}_{1,1x}(x_1) + a_{ijkl}(\mathbf{y})\, X^{11}_{k,ly}(\mathbf{y})\, u^{(0)}_{1,1x}(x_1) + a_{ij\beta\bar\beta}(\mathbf{y})\, s_\beta \varphi(x_1) \tag{3.13}$$

Integrating this expression over the PC P_1 (taking into account the fact that $\mathbf{u}^{(0)}, \varphi$ is independent of y), we obtain

$$\langle \sigma_{ij}^{(-4)} \rangle = \langle a_{ij11}(\mathbf{y}) + a_{ijkl}(\mathbf{y})\, X^{11}_{k,ly}(\mathbf{y}) \rangle\, u^{(0)}_{1,1x} + \langle a_{ij\beta\bar\beta} \rangle s_\beta \varphi \tag{3.14}$$

Let us put $j = 1$ in (3.14). Since the material of the beam is isotropic, it follows that $a_{i1\beta\bar\beta} = 0$ when $\beta\bar\beta = 23, 32$, and therefore the coefficient of φ in (3.14) is equal to zero (i.e. φ does not, in fact, appear in (3.14)). Then Eq.(3.1) when $m = -1$ will yield

$$(\langle a_{i111} + a_{i1kl}\, X^{11}_{k,ly} \rangle\, u^{(0)}_{1,1x}(x_1))_{,1x} = 0 \tag{3.15}$$

From the boundary condition $\mathbf{u}(\mathbf{x}) = 0$ at $x_1 = \pm 1$ and the asymptotic expansion (2.1), it follows that $u_1^{(0)}(\pm 1) = 0$. Eq.(3.15) with the above boundary condition (3.12) takes the form

$$u_1^{(1)} = -y_\alpha u^{(0)}_{\alpha,1x}(x_1) + V_1(x_1) \tag{3.16}$$

$$u_\beta^{(1)} = V_\beta(x_1) + s_\beta y_{\bar\beta} \varphi(x_1)\ (\alpha, \beta = 2, 3)$$

4°. Let us now take $k = 1$, $m = -3$ and $\mathbf{v} = \mathbf{v}^{(1)}(\mathbf{y})$. By analogy with 3° we have, for this case,

$$\sigma^{(-3)}_{ij,jy} = 0 \text{ and } \sigma^{(-3)}_{ij} n_j = 0 \quad \text{on } \Gamma_1 \tag{3.17}$$

When $m = -3$, Eq.(2.6) yields

$$\sigma_{ij}^{(-3)} = a_{ijkl}(\mathbf{y})\, u^{(2)}_{k,ly} + a_{ijk1}(\mathbf{y})\, u^{(1)}_{k,ly} \tag{3.18}$$

Substituting into (3.18) the expression for $\mathbf{u}^{(1)}$ (3.16) we obtain, taking into account (3.17),

$$(a_{ijkl}(\mathbf{y})\, u^{(2)}_{k,ly} + a_{ijk1}(\mathbf{y})\, V_{k,1x}(x_1) - a_{ij11}(\mathbf{y})\, y_\alpha u^{(0)}_{\alpha,1x1x}(x_1) + a_{ij\beta 1}(\mathbf{y})\, s_\beta y_{\bar\beta} \varphi_{,1x}(x_1))_{,jy} = 0 \text{ in } \Omega_1 \tag{3.19}$$

$$(a_{ijkl}(\mathbf{y})\, u^{(2)}_{k,ly} + a_{ijk1}(\mathbf{y})\, V_{k,1x}(x_1) - a_{ij11}(\mathbf{y})\, y_\alpha u^{(0)}_{\alpha,1x1x}(x_1) + a_{ij\beta 1}(\mathbf{y})\, s_\beta y_{\bar\beta} \varphi_{,1x}(x_1))\, n_j = 0 \text{ on } \Gamma_1 \tag{3.20}$$

and the condition that the function $\mathbf{u}^{(2)}(x_1, y)$ is periodic in y_1, with PC P_1.

In order to obtain the solution of the problem (3.19), (3.20), we introduce the functions $\mathbf{X}^{2\alpha}(\mathbf{y})$, which represent the solution of the following "second PC theory of beams of the first type"

$$(a_{ijkl}(\mathbf{y})\, X^{2\alpha}_{k,ly} - a_{ij11}(\mathbf{y})\, y_\alpha)_{,jy} = 0 \quad \text{in } P_1 \tag{3.21}$$

$$(a_{ijkl}(\mathbf{y})\, X^{2\alpha}_{k,ly} - a_{ij11}(\mathbf{y})\, y_\alpha)\, n_j = 0 \quad \text{on } \gamma$$

$\mathbf{X}^{2\alpha}(y)$ is periodic in y_1 with PC P_1, and the function $\mathbf{X}^3(\mathbf{y})$ represents the solution of the "second PC theory of beams of the second type"

$$(a_{ijkl}(\mathbf{y})\, X^3_{k,ly} + a_{ij\beta 1}(\mathbf{y})\, s_\beta y_{\bar\beta})_{,jy} = 0 \quad \text{in } P_1 \tag{3.22}$$

$$(a_{ijkl}(\mathbf{y})\, X^3_{k,\,yl} + a_{ij\beta 1}(\mathbf{y}) s_\beta y_{\bar\beta})\, n_j = 0 \quad \text{on} \quad \gamma$$

$\mathbf{X}^3(y)$ is periodic in y_1, with PC P_1.

Note 4. The problem of the second type has no analogue in the theory of plates /1, 8/, and is connected with torsion of the rod (beam).

Taking into account relations (3.9) and (3.21), we obtain

$$\mathbf{u}^{(2)} = \mathbf{X}^{1k}(\mathbf{y})\, V_{k,\,1x}(x_1) + \mathbf{X}^{2\alpha}(\mathbf{y})\, u^{(0)}_{\alpha,\,1x1x}(x_1) + \mathbf{X}^3(\mathbf{y})\, \varphi(x_1) \tag{3.23}$$

Taking into account Note 3, we can rewrite relations (3.23) in the following coordinate form:

$$u_1^{(2)} = X_1^{11}(\mathbf{y})\, V_{1,\,1x}(x_1) - y_\alpha V_{\alpha,\,1x}(x_1) + X_1^{2\alpha}(\mathbf{y})\, u^{(0)}_{\alpha,\,1x1x)}(x_1) + X_1^{3}(\mathbf{y})\varphi_{,\,1x}(x_1) \tag{3.24}$$

$$u_\beta^{(2)} = X_\beta^{11}(\mathbf{y})\, V_{1,\,1x}(x_1) + X_\beta^{2\alpha}(\mathbf{y})\, u^{(0)}_{\alpha,\,1x1x}(x_1) + X_\beta^{3}(\mathbf{y})\, \varphi_{,\,x}(x_1) \qquad (\alpha, \beta = 2, 3)$$

Substituting (3.24) into (2.6) for $m = -3$ and collecting terms, we obtain

$$\sigma_{ij}^{(-3)} = (a_{ij11}(\mathbf{y}) + a_{ijkl}(\mathbf{y})\, X^{11}_{k,\,ly}(\mathbf{y}))\, V_{1,\,1x}(x_1) + (-a_{ij11}(\mathbf{y})\, y_\alpha + a_{ijkl}(\mathbf{y})\, X^{2\alpha}_{k,\,ly}(\mathbf{y}))\, u^{(0)}_{\alpha,\,1x1x}(x_1) + a_{ijkl}(\mathbf{y})\, X^3_{k,\,ly}(\mathbf{y})\, \varphi_{,1x}(x_1) \tag{3.25}$$

Let us now integrate Eq.(3.25) over the PC P_1. Remembering that the functions $\mathbf{V}, \mathbf{u}^{(0)}$ are independent of $\mathbf{y}$, see (2.1) and (3.10), we obtain

$$\langle\sigma_{ij}^{(-3)}\rangle = \langle a_{ij11} + a_{ijkl} X^{11}_{k,\,ly}\rangle\, u_{1,\,1x} + \langle -a_{ij11} y_\alpha + a_{ijkl} X^{2\alpha}_{k,ly}\rangle\, u^{(0)}_{\alpha,\,1x1x} + \langle a_{ijkl} X^3_{k,\,ly}\rangle\, \varphi_{,1x} \tag{3.26}$$

Multiplying both sides of Eq.(3.25) by y_β and integrating over PC P_1 we obtain, for $j = 1$,

$$M_{\beta i}^{(-3)} = \langle y_\beta\,(a_{i111} + a_{i1kl} X^{11}_{k,\,ly})\rangle\, V_{1,\,1x} + \langle y_\beta\,(-a_{i111} y_\alpha + a_{i1kl} X^{2\alpha}_{k,ly})\rangle\, u^{(0)}_{\alpha,\,1x1x} + \langle y_\beta a_{i1kl} X^3_{k,\,ly}\rangle\, \varphi_{,1x} \tag{3.27}$$

From (3.5) we obtain, for $m = -3$, $i = 1$,

$$M^{(-3)}_{\alpha 1,\,1x} + \langle\sigma_{1\alpha}^{(-2)}\rangle = \langle g_1 y_\alpha\rangle_\gamma + \langle f_1 y_\alpha\rangle \tag{3.28}$$

and from (3.1) we have for $m = -2$, $i = \alpha$

$$\langle\sigma_{\alpha 1}^{(-2)}\rangle_{,1x} = \langle g_\alpha\rangle_\gamma + \langle f_\alpha\rangle \quad (\alpha = 2, 3) \tag{3.29}$$

Let us write

$$\begin{gathered} A_i^{\circ} = \langle a_{i111} + a_{i1kl} X^{11}_{k,\,ly}\rangle \\ A^1_{i\alpha} = \langle -a_{i111} y_\alpha + a_{i1kl} X^{2\alpha}_{k,\,ly}\rangle \\ {}^1A_{i\beta} = \langle y_\beta\,(a_{i111} + a_{i1kl} X^{11}_{k,\,ly})\rangle \\ A^2_{\alpha\beta i} = -\langle y_\beta\,(-a_{i111} y_\alpha + a_{i1kl} X^{2\alpha}_{k,\,ly})\rangle \\ B_{ij}^{\circ} = \langle a_{ijkl} X^3_{k,\,ly}\rangle, \quad B^1_{\beta i} = \langle y_\beta a_{i1kl} X^3_{k,\,ly}\rangle \quad (\alpha, \beta = 2, 3) \end{gathered} \tag{3.30}$$

We shall see later that the quantities (3.30) represent the elastic characteristics of the beam.

Let us consider Eq.(3.5) for $m = -3$, $i = \beta$ $(\beta = 2, 3)$. We have for this case $M^{(-3)}_{\alpha\beta,\,1x} + \langle\sigma_{\beta\alpha}^{(-2)}\rangle = \langle g_\beta y_\alpha\rangle_\gamma + \langle f_\beta y_\alpha\rangle$. Also, by definition, $M^{(-3)}_{\alpha\beta} = \langle y_\alpha \sigma_{\beta 1}^{(-3)}\rangle$. As a rule, in mechanics we consider not $M^{(-3)}_{\alpha\beta}$, but the quantity $M = \langle\sigma_{\beta 1}^{(-3)}\rangle s_\beta y_{\bar\beta}\rangle$ which represents the torsional moment. Since $\langle\sigma_{\beta 1}^{(-3)} s_\beta y_{\bar\beta}\rangle = M_{32}^{(-3)} - M_{23}^{(-3)}$, we obtain

$$M_{,\,1x} = \langle g_2 y_3\rangle_\gamma - \langle g_3 y_2\rangle_\gamma + \langle f_2 y_3\rangle - \langle f_3 y_2\rangle \tag{3.31}$$

Here we have used the relation $\langle\sigma_{23}^{(-2)}\rangle = \langle\sigma_{32}^{(-2)}\rangle$, which follows from the fact that $\sigma_{ij}^{(-2)} = \sigma_{ji}^{(-2)}$ /5/.

4. Limit problem. Let us collect together all relations obtained above (with the source of each relation quoted on the left-hand side of the relation)

$$(3.26),\quad i = j = 1:\quad \langle\sigma_{11}^{(-3)}\rangle = A_1^{\circ} V_{1,\,1x} - A^1_{1\alpha} u^{(0)}_{\alpha,\,1x1x} + B_{11}^{\circ}\varphi_{,1x} \tag{4.1}$$

$$(3.1),\quad i = j = 1,\ m = -3:\quad \langle\sigma_{11}^{(-3)}\rangle_{,1x} = 0 \tag{4.2}$$

$$(3.27),\quad i = 1:\quad M_{\beta 1}^{(-3)} = {}^1A_{1\beta} V_{1,\,1x} - A^2_{\alpha\beta 1} u^{(0)}_{\alpha,\,1x1x} + B^1_{\beta 1}\varphi_{,1x} \tag{4.3}$$

$$(3.28),\quad -M^{(-3)}_{\alpha 1,1x} + \langle \sigma^{(-2)}_{1\alpha} \rangle = \langle g_1 y_\alpha \rangle_\gamma + \langle f_1 y_\alpha \rangle \tag{4.4}$$

$$(3.29),\quad \langle \sigma^{(-2)}_{\beta 1} \rangle_{,1x} = \langle g_\beta \rangle_\gamma + \langle f_\beta \rangle \tag{4.5}$$

$$(3.27),\quad \beta i = 23, 32, m = -3:\quad M = ({}^1A_{23} - {}^1A_{32}) V_{1,1x} - (A^2_{\alpha 32} - A^2_{\alpha 23}) u^{(0)}_{\alpha,1x1x} + (B_{32}{}^1 - B_{23}{}^1)\varphi_{,1x} \tag{4.6}$$

$$(3.31),\quad M_{,1x} = \langle g_2 y_3 \rangle_\gamma - \langle g_3 y_2 \rangle_\gamma + \langle f_2 y_3 \rangle - \langle f_3 y_2 \rangle \tag{4.7}$$

$$(\alpha, \beta = 2, 3)$$

The resulting limit system represents a problem in the theory of beams. Here $V_{1,1x}$ is the axial deformation, $u^{(0)}_{\alpha,1x1x}$ are the curvatures and φ is the angle of torsion. The relations (4.2), (4.4), (4.5) and (4.7) represent the equations of equilibrium, and (4.1), (4.3) and (4.6) the defining relations of the beam. The coefficients of the latter are found from the solution of CP (3.9), (3.21) and (3.22) representing three-dimensional problems of the theory of elasticity with boundary conditions of special type.

Let us consider the boundary conditions. From the asymptotic expansion (2.1) and boundary condition $\mathbf{u}(\mathbf{x}) = 0$ we have, for $x_1 = \pm 1$,

$$\mathbf{u}^{(0)}(\pm 1) = 0,\quad \mathbf{u}^{(1)}(\pm 1, \mathbf{y}) = 0 \tag{4.8}$$

Substituting expressions (3.16) into (4.8), we obtain

$$-y_\alpha u^{(0)}_{\alpha,1x}(\pm 1) + V_1(\pm 1) = 0,\quad V_\beta(\pm 1) + s_\beta y_{\bar\beta}\varphi(\pm 1) = 0$$

for all y_2, y_3 $(\beta = 2, 3)$, and from this it follows that

$$V_1(\pm 1) = 0,\quad u_\alpha^{(0)}(\pm 1) = 0,\quad u^{(0)}_{\alpha,1x}(\pm 1) = 0,\quad \varphi(\pm 1) = 0 \tag{4.9}$$
$$(\alpha = 2, 3)$$

Relations (4.9) represent the covering set of boundary conditions for problem (4.1)-(4.7).

5. *Example. A cylindrical rod.* *First CP.* Let the region Ω_ε be a cylinder $[-1, 1] \times S_\varepsilon$ and let the elastic constants of the material $a_{ijkl}(\mathbf{y})$ depend only on the variable y_2, y_3 (but not on y_1). In this case the first local problem will have a solution of the type

$$X_1{}^{11}(\mathbf{y}) = 0,\quad X_\beta{}^{11}(\mathbf{y}) = X_\beta{}^{11}(y_2, y_3),\quad (\beta = 2, 3) \tag{5.1}$$

It is clear that the functions (5.1) are periodic in y_1 on PC P_1. Substituting (5.1) into (3.9), we obtain a two-dimensional problem in terms of the functions

$$(a_{\alpha\beta\gamma\delta} X^{11}_{\gamma,\delta} + a_{\alpha\beta 11})_{,\beta} = 0 \quad \text{in } S_1 \tag{5.2}$$
$$(a_{\alpha\beta\gamma\delta} X^{11}_{\gamma,\delta} + a_{\alpha\beta 11})\, n_\beta{}^1 = 0 \quad \text{on } \partial S_1$$
$$(\alpha, \beta, \gamma, \delta = 2, 3),\ S_1 = \varepsilon^{-1} S_\varepsilon$$

where ($\mathbf{n}^1$ is the outer normal to ∂S_1). Moreover, if the material of the beam is homogeneous, the solution of problem (5.2) can be found in explicit form $X_\beta{}^{11} = W_\beta y_\beta$ (no summation over β). The constants $\{W_\beta\}$ are found by substituting the above expression into (5.2), and in terms of the Lamé constants λ, μ /5/ they are $W_2 = W_3 = \lambda/2(\lambda + 2\mu)$.

The tensile stiffness of the rod will, in this case, be equal to

$$A_1{}^0 = \frac{(3\lambda + 2\mu)\mu}{\lambda + 2\mu} \operatorname{mes} S_\varepsilon$$

where $\operatorname{mes} S_\varepsilon$ is the area of S_ε (the area of transverse cross-section of the rod).

Second CP of the first type. We shall seek the solution of (3.21) in the form

$$X_1^{2\alpha}(\mathbf{y}) = 0,\quad X_\beta^{2\alpha}(\mathbf{y}) = X_\beta^{2\alpha}(y_2, y_3)\quad (\alpha, \beta = 2, 3)$$

It is clear that the function is periodic in y_1. The equation from (3.21) for this function (at $i = 1$) takes the form

$$(a_{1jkl} X^{2\alpha}_{k,ly} - a_{1j11} y_\alpha)_{,jy} = (a_{1\beta\gamma\delta} X^{2\alpha}_{\gamma,\delta} - a_{1\beta 11} y_\alpha)_{,\beta} = 0$$

The last equation holds for isotropic materials /5/ by virtue of $a_{1\alpha\gamma\delta} = 0$, $a_{1\beta 11} = 0$ when $\beta, \gamma, \delta = 2,3$ The first relation from the boundary conditions (3.21) also holds (it should be noted that the first coordinate of the normal to γ, in the case in question, is equal to zero). As a result we obtain the following two-dimensional problem:

$$(a_{\varkappa\beta\gamma\delta} X^{2\alpha}_{\gamma,\delta} - a_{\varkappa\beta 11} y_\alpha)_{,\beta} = 0 \quad \text{in } S_1 \tag{5.3}$$

$$(a_{\varkappa\beta\gamma\delta} X^{2\alpha}_{\gamma,\delta} - a_{\varkappa\beta 11} y_\alpha)\, n_\beta{}^1 = 0 \quad \text{on } \partial S_1$$

The flexural stiffness of the beam is given by the expression

$$A^2_{\varkappa\beta 1} = \langle y_\beta (\alpha_{1111} y_\varkappa - \alpha_{11\gamma\gamma} X^{2\varkappa}_{\gamma,\gamma})\rangle \quad (\varkappa, \beta, \gamma = 2, 3)$$

In particular, the flexural stiffness in the plane Ox_1x_2 is

$$A^2_{221} = \langle a_{1111} y_2^2\rangle - \langle y_2 a_{11\gamma\gamma} X^{21}_{\gamma,\gamma}\rangle \quad (\gamma = 2, 3) \tag{5.4}$$

The first term on the right-hand side of relation (5.4) represents the flexural stiffness of the beam calculated in accordance with engineering theory /5/. We see that the second term on the right-hand side of (5.4) is, in general, non-zero (since $X^{21}(y_2, y_3) = 0$ is not, in the general case, a solution of problem (5.3)).

Second CP of the second type. We shall seek a solution of problem (3.22) in the form

$$X^3(y) = X_1^3(y_2, y_3), \quad X_\beta^3(y) = 0 \quad (\beta = 2, 3) \tag{5.5}$$

The periodicity of the function (5.5) in y_1 is obvious. Substituting relations (5.5) into (3.22), we obtain

$$(a_{i\delta 1\gamma} X^3_{1,\gamma} + a_{i\delta\beta 1} s_\beta y_{\bar\beta})_{,\delta} \tag{5.6}$$

Since $a_{i\delta 1\gamma} = 0$, $a_{i\delta\beta 1} = 0$ when $i, \delta, \beta, \gamma = 2{,}3$ for isotropic materials, it follows that there remains, from (5.6), only a single non-trivial equation (when $i = 1$) for $X_1^3(y_2, y_3)$:

$$(a_{1\delta 1\gamma} X^3_{1,\gamma} + a_{1\delta\beta 1} s_\beta y_{\bar\beta})_{,\delta} = 0 \text{ in } S_1$$

Remembering that in this case $n_1 = 0$, we obtain the corresponding boundary condition

$$(a_{1\delta 1\gamma} X^3_{1,\gamma} + a_{1\delta\beta 1} s_\beta y_{\bar\beta}) n_\delta^1 = 0 \quad \text{on} \quad \partial S_1$$

Note 5. In the case of frameworks and similar, highly porous constructions, we can use the method from /9-13/ to solve the CP.

6. On the validation of the asymptotic expressions. Using the technique of constructing the asymptotic formulas analogous to that used in /1/, we can validate the results in the same manner as in /1/. Let us introduce, in the region Ω_1, the function (the normalized displacement field)

$$U_i^\varepsilon(z) = \begin{cases} \varepsilon^{-1} u_1(z) & \text{when } i = 1, \\ u_i(z) & \text{when } i = 2, 3, \quad z = (x_1, y_2, y_3) \in \Omega_1 \end{cases}$$

where (u_1, u_2, u_3) is the solution of problem (1.1), (1.2) in the variables z, and consider the asymptotic expression for the displacement field (2.1) with an accuracy up to terms of order ε:

$$u_i(z) = \begin{cases} 0 + \varepsilon (V_1 - y_\alpha u^{(0)}_{\alpha,1x}) + \ldots & \text{when } i = 1 \\ u_i^{(0)} + \varepsilon(\ldots) + \ldots & \text{when } i = 2, 3 \end{cases} \tag{6.1}$$

Let us introduce the normal displacement field corresponding to (6.1)

$$U_i(z) = \begin{cases} \varepsilon^{-1} u_1 = V_1 - y_\alpha u^{(0)}_{\alpha,1x} & \text{when} \quad i = 1 \\ u_i^{(0)} & \text{when} \quad i = 2, 3 \end{cases}$$

Proposition. $U^\varepsilon \to U$ is weak in $V(\Omega_1)$ as $\varepsilon \to 0$.

The proof is obtained by reproducing (with the technical corrections arising from changing the dimensions of the problem from three to one and not from three to two as in /1/) the arguments of Sect.3 (a priori estimates) and 6.3 (proof of the convergence) of /1/.

Note 6. The proposition leads to the assertion on convergence, without taking into account the torsion (the terms corresponding to torsion of the rod are of higher order than those corresponding to normal flexures). The validation of the asymptotic formulas taking the torsion into account (e.g. when $u_\alpha^{(0)} = 0$) is not the same as in /1/ (since there is no torsion in the plates discussed in /1/).

Note 7. For a cylindrical rod of coaxial construction the problem was studied in detail in /14/. The asymptotic formulas given above for these rods are identical with those used in /14/.

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